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ON A PARTICULAR INSCRIBED RECTANGLE OF A CONIC *

BY

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It is well-known that the inscribed rectangles of a conic play an important part in the theory of its normals.† In the binary representation of the conic the inscribed rectangles are given by the syzygetic pencil of quartics determined by the remarkable sextic representing the extremities of the axes and the points at infinity. The particular inscribed rectangle whose vertices are the four points of contact of the circular tangents, or alternatively whose sides are the four directrices, is of special interest. The object of this paper is to exhibit the significance of this rectangle in the metrical geometry of the conic.

1. The conic and its evolute

Since the evolute is a curve of order six and class four, there are twelve centres of curvature on the conic, and eight common tangents of the conic and its evolute. But a tangent to the evolute is a normal to the conic, and if it is also a tangent to the conic it must be a circular tangent. But there are only four circular tangents to the conic and hence the eight common tangents of the conic and its evolute coincide in pairs with the four circular tangents. From this, since the evolute is the envelope of the normals, it follows that the evolute touches the conic at the points of contact of the circular tangents. Therefore eight of the twelve centres of curvature which lie on the conic coincide in pairs with the vertices of the special rectangle we are considering. The other four centres of curvature which lie on the conic may be called the residual centres of curvature. These and their bases on the conic have been specified below,

* My thanks are due to Dr. Vaidyanathaswami who has helped me much in the preparation of this paper.

† Vaidyanathaswami, "On the feet of concurrent normals of a conic." *Jour. Ind. Math. Soc.*, Vol. 18 (1930-31) page 296.

both metrically and in terms of the covariants of the special rectangle under consideration.

2. The normal correspondence

Let $f = a_x^4$ represent parametrically the points of contact of the circular tangents to a conic S. Then from Meyer's "transference principle" the ∞^3 conics outpolar to the confocal system generated by S cut out on S tetrads which are apolar to f. These ∞^3 conics are rectangular hyperbolas, since the circular points constitute one of the confocals of S. Hence,

The necessary and sufficient condition that four points t_1 t_2, t_3, t_4 on S should lie on an outpolar rectangular hyperbola is that

$$a_{t_1}, a_{t_2}, a_{t_3}, a_{t_4} = 0 \qquad \ldots \quad (2{\cdot}1)$$

In particular the tangent and normal at a point being conjugate lines at the right angles constitute an outpolar rectangular hyperbola. Hence if the normal at x meets the conic again in y we have from (2·1) $a_x^3 a_y = 0$. i.e. *the normal at a point x meets the conic again at the linear polar of x with respect to f.*

3. The Hessian of f

It follows from the last result above that the three normals which pass through a point y on the conic (other than that at y) are given by $a_y a_x^3 = 0$, i.e. by the first polar of y with respect to f. By varying y we obtain a pencil of cubics each of which gives three points on the conic such that the normals thereat concur at a point on the conic. If a member of this pencil has a double factor we get two consecutive normals intersecting on the conic and hence the double factor gives a point the centre of curvature at which lies on the conic. But the double factors are the factors of the Jacobian of the pencil which in this case is the hessian of f. Hence

The bases of the four residual centres of curvature are the vertices of the rectangle given by the hessian of f.

This theorem may also be stated as follows, from the known relation between a tetrad and its hessian on the conic.

Let Γ be the unique inpolar confocal to S. Then Γ intersects S in a rectangle whose vertices are precisely the bases of the four residual centres of curvature on the conic. (3·1)

4. The Steinerian of f.

To find the four residual centres of curvature on the conic, it may be observed that if y is one of them, the first polar of y with respect to f has a double factor, i.e. its discriminant is zero. The condition for this is

$$(ab)^2(cd)^2(ac)(bd)a_y b_y c_y d_y = 0.$$

This reduces to $\frac{1}{3} jf - \frac{1}{2} iH = 0$; which is called the Steinerian of f. Hence

The Steinerian of f gives the inscribed rectangle of the conic, whose vertices are the four residual centres of curvature on the conic.

A metrical specification of the Steinerian points is given in § 6.

5. The Cohessian of f

First, the following theorem may be proved:—

If f is any binary quartic and four points ultimately coincident with a point x are equicross with their linear polars with respect to f, then x is a zero of the co-hessian of f*

Proof:—If y_1, y_2, y_3, y_4 be equicross with their linear polars with respect to $f = a_x^4 = b_x^4 = c_x^4 = d_x^4$, we have

$$\frac{(y_1 y_2)(y_3 y_4)}{(y_1 y_4)(y_3 y_2)} = \frac{(ab)\, a_{y_1}^3\, b_{y_2}^3 \cdot (cd)\, c_{y_3}^3\, d_{y_4}^3}{(ad)\, a_{y_1}^3\, d_{y_4}^3 \cdot (cb)\, c_{y_3}^3\, b_{y_2}^3} \qquad \ldots \quad (1)$$

But by Gordan's Theorem,

$$(ab)\, a_{y_1}^3\, b_{y_2}^3 = \{\tfrac{3}{2}\, [(ab)^2\, a_{y_1}^2\, b_{y_1}^2]_{y_2}^2 + \tfrac{1}{4}\, i\, (y_1\, y_2)^2\}\, (y_1\, y_2)$$

Expanding similarly all the terms on the right hand side of (1) and cancelling common factors on both sides, the equation (1) becomes

$$\{\tfrac{3}{2}\, H_{y_1^2\, y_2^2} + \tfrac{1}{4}\, i\, (y_1\, y_2)^2\}\{\tfrac{3}{2}\, H_{y_3^2\, y_4^2} + \tfrac{1}{4}\, i\, (y_3\, y_4)^2\} -$$
$$\{\tfrac{3}{2}\, H_{y_1^2\, y_4^2} + \tfrac{1}{4}\, i\, (y_1\, y_4)^2\}\{\tfrac{3}{2}\, H_{y_3^2\, y_2^2} + \tfrac{1}{4}\, i\, (y_3\, y_2)^2\} = 0$$

where H is the hessian of f.

Putting $y_1 = y_2 = x$ and $y_3 = y_4 = y$ in this we get

$$\tfrac{9}{4}\, H_x^4\, H_y^4 - \{\tfrac{3}{2}\, H_{x^2\, y^2} + \tfrac{1}{4}\, i\, (xy)^2\}^2 = 0 \qquad \ldots \quad (2)$$

The left-hand member of (2) is a double binary form in x and y and may be expanded in powers of (xy), i.e. in the form

$$A + B\, (xy) + C\, (xy)^2 + \cdots$$

* If f is a binary n-ic, the corresponding covariant is $2\,(n-2)\, jf - ni$ R, where j is the second wronskian, i the fourth transvectant and h the hessian of f. The method of proof is the same as for the quartic.

Now $A=0$, since the expression vanishes when $x=y$. Also the expression is symmetrical with respect to x and y and therefore $B=0$. If C is not zero we may divide out the expression by $(xy)^2$ which is not zero and the quotient will tend to C as $y \to x$. But C is the same but for a numerical factor, as the result got by operating on the expression with Ω^2 (Ω being the differential operator $\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)$ and then putting $y=x$ in the resulting expression. That result is found on reduction to be $126\, jf - 126\, i\mathrm{H}$, where j and i are the two invariants of f. Therefore the required covariant is $jf - i\mathrm{H}$, which is the co-hessian of f.

Now we have already proved that the normal at a point x intersects the conic again in the linear polar of x with respect to f. Also the lines joining corresponding points of two homographic ranges on the conic envelope a conic having double contact with the given conic. It is also known that the cohessian is the only member of the syzygetic pencil apolar to f and that it has the same hessian as that of f. Putting these facts together we may derive the following theorem.

Let Γ' be the inpolar conic through the hessian points, other than Γ of (3·1.) Then the contact tetrad of Γ' with S (viz the cohessian tetrad of f) has the properties :—

(1) It is the unique inscribed rectangle of S, such that an outpolar rectangular hyperbola can be drawn through its vertices.

(2) Each point P of the tetrad is such that there is a conic having double contact with S and having four point contact with the evolute of S at the centre of curvature at P.

6. The apolar triads of f

Since f is a rectangle an apolar triad of f is a pedal triad (i.e. the normals at the three points are concurrent *).

Further from 2·1 PQR is a triad apolar to f if and only if every outpolar conic through P, Q, R is a rectangular hyperbola, or what is the same thing, if the centre of perspective of PQR and its polar triangle is the orthocentre of PQR.

Now the triangles apolar to f form a pencil of inscribed triangles in S and therefore circumscribe a unique conic S′. A particular triangle apolar to f is one whose vertices are two coincident points

* Vaidyanathaswami loc. cit.

at a hessian point and the corresponding steinerian point, for this triangle is such that the line joining each vertex to the pole of the opposite side is perpendicular to that side. Therefore the conic S′ touches the tangents and normals at the hessian points and also touches the evolute of S at the steinerian points. But the tangents and normals to S at the hessian points are respectively the tangents to S and Γ, at their points of intersection, (since confocal conics cut at right angles). Therefore S′ is none other than the ϕ - conic of S and Γ.

Incidentally we have proved the theorem :

The four residual centres of curvature on the conic S, namely the steinerian points of f, are the points of intersection of S and the ϕ - conic of S and its inpolar confocal.

7. A (2, 2) Correspondence on the conic

It is a consequence of (2·1) that the necessary and sufficient condition that an outpolar rectangular hyperbola should touch the conic S at the two points x and y is that $a_x^2 a_y^2 = 0$.

Now the polar correspondence given by $a_x^2 a_y^2$ is the correspondence determined by the polarity of the unique inpolar confocal Γ, considered as an envelope.

Also the discriminant of $a_x^2 a_y^2$ considered as a quadratic is x or y in the hessian of f.

Hence we have the following theorem.

Given a point P on the conic S, there are two outpolar rectangular hyperbolas touching S at P and having double contact with S, their other points of contact with S being the points of contact of the two tangents of S which are conjugate with regard to Γ, to the tangent at P to S. When P is the point of contact of a circular tangent, the two outpolar rectangular hyperbolas coincide with the circular tangent taken twice. When P is a residual centre of curvature the two outpolar rectangular hyperbolas coincide into one touching S at P and its base on the conic.

THE LIMITING POINTS OF THE SET OF THE UPPER AND THE LOWER SUMS IN RIEMANN INTEGRATION

BY

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§ 1. $f(x)$ being a function bounded in $a \leqslant x \leqslant b$, the set E of the upper sums $\sum M_r \delta_r$, and e of the lower sums $\sum m_r \delta_r$, for all modes of subdividing (a, b) into abutting subintervals, are bounded sets.* In this paper I consider the limiting points of these sets and describe the mode of subdividing (a, b) so as to give an upper sum (or a lower sum as the case may be), which differs from a given limiting point by an arbitrarily assigned positive number ε. The lower limit U of the set E and the upper limit L of the set e are considered in Darboux's theorem, which gives subdivisions of (a, b), whose upper sums are $< U + \varepsilon$ and lower sums $> L - \varepsilon$; to get such sums the mode of subdivision is merely to make the maximum sub-interval less than a certain η.

§ 2. THEOREM: *If M and m be the upper and the lower bounds, U and L the upper and the lower Integrals of $f(x)$ in (a, b), then every point between U and M $(b-a)$ is a limiting point of the set E of the upper sums; and similarly, every point between L and m $(b-a)$ is a limiting point of the set e of the lower sums.*

Proof: Let P be a number between U and Mh and $P = Kh$, where $h = (b-a)$. Now there exists a number μ, $M \geqslant \mu \geqslant m$, such that $\mu h = U$. Clearly $M \geqslant K \geqslant \mu \geqslant m$. Consider $M(x-a) - \overline{\int_a^x} f(x)dx$. Since this is a continuous function of x, whose maximum and minimum values in (a, b) are $(M-\mu)h$ and zero, therefore there exists a value c of x such that

$$M(c-a) - \overline{\int_a^c} f(x)dx = (K-\mu)h.$$

Let $\overline{\int_a^c} f(x)dx = A$. Then $\overline{\int_c^b} f(x)dx = \mu h - A$. Now (it is easily proved by the method of continued bisection that) there always exists a point ξ in (a, b) such that every neighbourhood of ξ has M for upper bound.

First let us prove our theorem taking ξ to coincide with a. Then in any subdivision of (a, b) in which (a, c) is a sub-interval, the

* M_r and m_r are the upper and the lower bounds of $f(x)$ in the sub-interval δ_r.

contribution of (a, c) to the upper sum is equal to $A+(K-\mu)h$. And given $\varepsilon>0$ arbitrarily, we can, by Darboux's theorem, find an $\eta>0$ such that when (c, b) is subdivided into intervals of length less than η each, the upper sum lies between $\mu h-A$ and $\mu h-A+\varepsilon$. Every such sub-division of (c, b) together with the interval (a, c) will therefore constitute a sub-division of (a, b) whose upper sum lies between $A+(K-\mu)h+(\mu h-A)$ and $A+(K-\mu)h+(\mu h-A)+\varepsilon$ i.e. between P and $P+\varepsilon$.

But (c, b) can be sub-divided in an infinity of ways such that the maximum sub-interval is less than η, and we have only to add the interval (a, c) to each of these subdivisions of (c, b) to obtain a sub-division of (a, b). Therefore there exist an infinity of upper sums between P and $P+\varepsilon$. Hence P is a limiting point of the set E.

We have taken $a=\xi$ in the above proof for the sake of convenience. But even if $\xi\neq a$, we can, since the upper integral is a continuous function of the upper limit of integration, still obtain an interval (a_1, c_1) in (a, b) so that ξ is an interior point (or end point according to circumstances) of (a_1, c_1) and at the same time

$$M(c_1-a_1)-\overline{\int_{a_1}^{c_1}} f(x)dx=(K-\mu)h.$$

Then in the above proof we have only to replace the interval (a, c) by the interval (a_1, c_1) and (c, b) by the intervals (a, a_1), (c_1, b).

In the case of the set e we should take $\underline{\int_{a}^{x}} f(x)dx-m(x-a)$ instead of $M(x-a)-\overline{\int_{a}^{x}} f(x)dx$ and the interval (a_1, c_1) so that the lower bound of $f(x)$ in it is m.

§ 3. Consider for instance the function $f(x)$ defined in (0,1) by $f(x)=x$ when x is irrational, and $f(x)=\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}$ when $x=\frac{p}{q}$, and $f(0)=0$. The discontinuities of this function are enumerable since they occur at rational points (except 0 and 1*). The function is therefore integrable and the value of the integral is easily seen to be $\frac{1}{2}$ by a cosideration of the upper or the lower sums. We have $h=1$, $\mu=\frac{1}{2}$, $U=L=\frac{1}{2}$, $M=1$, $m=0$, $\xi=1$ (M is attained at ξ). All points between $\frac{1}{2}$ and 1 are limiting points of E and all points between 0 and $\frac{1}{2}$ are limiting points of e. For, suppose we wanted a subdivision of (0,1) whose upper sum differs from the number $\frac{3}{4}$ by a given quantity. Then $(K-\mu)\ h=\frac{1}{4}$ and since $f(1)=M$, therefore we must find a point a_1 such that $\int_{a_1}^{1}[1-f(x)]\ dx=\frac{1}{4}$. But $\int_{a_1}^{1} f(x)\ dx=(1-a_1)(1+a_1)/2$. Hence

* The continuity at $x=1$ is omitted in Hardy's *Pure Mathematics* Ex. 21. P. 179, 4th Edition.

a_1 should satisfy the equation $2a_1^2-4a_1+1=0$. So, $a_1=\left(1-\frac{1}{\sqrt{2}}\right)$. Thus (a_1, c_1) is the interval $\left(1-\frac{1}{\sqrt{2}}, 1\right)$. To subdivide $\left(0, 1-\frac{1}{\sqrt{2}}\right)$ we first observe that $\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}-\frac{p}{q}<\varepsilon$ i. e.

$$q^2\ (\varepsilon^2\ q^2+\varepsilon^2-1)+p\ (2\varepsilon\ q^3+2\varepsilon q+p)>0,$$

if $(\varepsilon^2q^2+\varepsilon^2-1)>0$ (for all $p<q$); and $(\varepsilon^2q^2+\varepsilon^2-1)>0$ if $q>1/\varepsilon$. Thus when $q>1/\varepsilon$, we have $\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}-\frac{p}{q}<\varepsilon$.

Now divide $\left(0, 1-\frac{1}{\sqrt{2}}\right)$ into n^3 intervals of length $\frac{1}{n^3}\left(1-\frac{1}{\sqrt{2}}\right)$ each, where $n>1/\varepsilon$. Mark off the points for which $n\leqslant 1/\varepsilon$. Such points are not more than n^2 in number and are situated in not more than n^2 of the n^3 sub-intervals. The contribution S_1 of the n^2 intervals to the upper sum is less than

$$n^2.\ 1.\ \frac{1}{n^3}\left(1-\frac{1}{\sqrt{2}}\right)<\left(1-\frac{1}{\sqrt{2}}\right)\varepsilon, \qquad \ldots \text{(i)}$$

since the maximum of $f(x)$ is $\leqslant 1$. The contribution S_2 of the remaining (n^3-n^2) intervals is less than

$$\frac{1}{n^3}\left(1-\frac{1}{\sqrt{2}}\right)\sum_{\gamma=1}^{n^3}\left[\frac{\gamma\left(1-\frac{1}{\sqrt{2}}\right)}{n^3}+\varepsilon\right]; \qquad \ldots \text{(ii)}$$

for, in the interval $\left[\frac{(\gamma-1)\left(1-\frac{1}{\sqrt{2}}\right)}{n^3}, \frac{\gamma\left(1-\frac{1}{\sqrt{2}}\right)}{n^3}\right]$ the maximum of $f(x)$ is less than $\left[\frac{\gamma\left(1-\frac{1}{\sqrt{2}}\right)}{n^3}+\varepsilon\right]$. In forming (ii) we have, for the sake of convenience, repeated the n^2 intervals each multiplied by the corresponding $\frac{\gamma\left(1-\frac{1}{\sqrt{2}}\right)}{n^3}+\varepsilon$. The effect has been merely to increase the quantity than which S_2 is less.

Now the left member of (ii) is equal to

$$\left(1-\frac{1}{\sqrt{2}}\right)^2\left(\varepsilon+\frac{n^3+1}{2n^3}\right)<\left(1-\frac{1}{\sqrt{2}}\right)^2\left(\tfrac{1}{2}+\varepsilon+\frac{\varepsilon^3}{2}\right).$$

Hence from (i), $S_1+S_2<\frac{3}{4}+\varepsilon$. Similarly we could take any number between 0 and $\frac{1}{2}$ and show it to be a limiting point of the set e.

A NOTE ON FREQUENCY DISTRIBUTION IN SERIES

By

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For a random variable that can only assume discrete values, the method of generating functions is in common use as an aid to formulating moments.[1] The probability of the value x_j being p_j, the probability generating function is defined by $\sum p_j z^{x_j}$, and moments, factorial moments, and semi-invariants are obtained as coefficients in the various power-series expansions if they exist, of the function after substitution of e^a, $1+a$, for the parameter z, the third being obtained by expanding the logarithm of the function after the first change of variable. But here, the parameter is introduced "for the sole purpose of preventing the terms from being merged together",[2] and is, so to speak, without any value of its own.

I suggest, in what follows, a similar method of generating the probabilities and the moments, but making use of the variable z as an essential part of the distribution. The difference is that one obtains formulae involving the differential calculus in place of algebra, and a connection is to be seen at once with the formal apparatus of the theory of functions of a complex variable, summability, and Tauberian theorems.

1. Let $f(z)$ be an analytic function in the neighbourhood of the origin, with real non-negative derivatives at the origin. That is, $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$, a_i real, $\geqslant 0$ for all i, and let the series converge to f for some region beyond just the origin itself, say $|z| < r$. Then, we shall say that f gives a distribution, the probability of the value n being given by $a_n z^n/f$. For the present, the variate x is assumed to take on only the values $0, 1, 2, 3, \cdots n \cdots$

The kth moment, the expectation of n^k, is given by $\frac{\theta^k f}{f}$, where θ is the operator $z\frac{d}{dz}$ and the exponent k indicates a k-fold application. The corresponding factorial moment is given by $\frac{z^k d^k f}{f\, dz^k}$. The operator θ and the occurrence of the factor $1/f$ suggest

immediately, the transformation $\phi = \log f$, $s = \log z$. Then the successive derivatives of $\phi(s)$ are the semi-invariants λ_i; $\phi' = m$, $\phi'' = \sigma^2$ etc.

By merely specifying the function, various distributions can be characterized. The Bernoulli distribution is given by $f = [1+z]^n$. In the usual terminology, $p = z/(1+z)$, $q = 1/(1+z)$ gives the probabilities for success and failure in a single trial. The mean value is $m = zf' / f = nz/(1+z)$ $1 = np$, the variance being $\sigma^2 = nz/(1+z)^2 = npq$. Here, as in general, we shall take only real non-negative values of z for statistical purposes.

For the Poisson distribution, $f = e^z$, $m = \sigma^2 = \lambda_i = z$. Distributions having various properties can be built up by solving the corresponding differential equations. But the use of $\phi(s)$ shows that a non-vanishing multiplicative arbitrary constant in f as well as in z can be absorbed, or ignored, without any essential change. Thus, $\sigma^2 = km$ leads to $f = e^{z^k}$. For $\sigma^2 = m - k$ we have $f = z^k e^z$. The first would correspond to a Poisson distribution of twins, triplets, etc. for $k = 2, 3, \ldots$; the second is essentially the generalised Poisson distribution for a variate that cannot take any value less than k. But a variate occurring in even or odd values only where m tends to equality with σ^2 as both increase would be distributed, say, according to cosh z, or sinh z.

2. The direct determination of f would be useful for some properties. The distribution of a sum of two variates distributed according to $A(z) = a_0 + a_1 z + \cdot\cdot$ and $B(t) = b_0 + b_1 t + \cdot\cdot$ would be given by the probability for the value n of the sum equal to $\sum_{i+j=n} a_i z^i b_j t^j / AB$. For two Bernoulli distributions with the same n the function is $(1+z+t+zt)^n$, corresponding to a new parameter $z+t+zt$, or to the formula for compound probability $p = p_1 + p_2 - p_1 p_2 = p_1 q_2 + p_2 q_1 + p_1 p_2$. The Poisson distribution is essentially the only one where the sum is distributed according to the same law with the new parameter a sum of the two original parameters: $f(z) f(t) = f(z+t)$.

In some observations, even for continuous distributions, the quantities ϕ and s are observed directly, not f and z. Fechner's law naturally comes to mind here. One example that suggests itself is that of photo-sensitometry[3], where the logarithm of the darkening is plotted against the logarithm of the exposure for a given emulsion and time of development. The "gamma" value usually published is merely the slope of the curve, i.e. $\phi'(s)$ for the central portion,

where the curve has a point of inflection, $\phi''=0$, and approximates very closely to its tangent. According to our theory, this is a mean value, and in fact, if the darkening depends in a certain way upon the number of elementary particles affected (the grains of the emulsion), this interpretation would be entirely accurate. The curve of sensitivity usually published looks uncommonly like half a Gaussian curve of normal frequency, but this cannot be verified unless very accurate measurements are made in the range of solarization; these last have been neglected, because they appear to have no practical use in photography, the useful portion of the curve being only in the neighbourhood of the point of inflection.

The transformation to s, ϕ, suggests that for a denumerable set of discrete values of the variable, the proper generalisation of our distribution is not in the form of power series with fractional exponents, but Dirichlet series for f or ϕ. In the usual notation, these would be given, substituting $-s$ for s by $\sum a_n e^{-\lambda_n s}$. The connection with the theory of numbers is obvious, and need not be considered in detail here.

3. Another obvious connection with analysis is the notion of summability given by each of our distributions. The Poisson distribution, for instance, leads to Borel summability. Let us interpret n as the number of observations, or the number attached to an observation or measurement, of some quantity s; let $s_1, s_2, s_3 \cdots$ be the successive values obtained in this way. The probability of n observations (or the nth observation) having taken place being distributed in a Poisson series, the expectation of s is $\sum e^{-z} s_n z^n/n!$ The question: when are we justified in taking the limit of this as $z \longrightarrow \infty$ as the limit of s_n, amounts to the Tauberian problem: when does a sequence summable-B converge? A better question would be: when can we take the limiting expectation as limit of the arithmetic means, that is when is B-summability equivalent to (C, 1) summability? Inasmuch as, in practice, the limit can never be attained, the important question is not only the approach to a limit, but actually the behaviour and distribution in the vicinity of the limit. That is, the Tauberian idea must be generalized. In view of the fact that a connection is known between Tauberian theorems and probability [5] through the Fourier-Stieltjes integral, I shall merely point out the phenomenon here.

4. Of the infinite series given, there naturally exist two classes: those with a finite and those with an infinite radius of convergence. For the first, we can take the radius as unity, since

a factor in the parameter z makes no real difference. The second class contains polynomials and integral functions, and is more interesting in a way because the distributions can be made to tend to the normal, under certain restrictions. For a polynomial, this is a matter of adjusting n, the degree ($=\lim. zf'/f$) and the parameter z carefully, letting n tend to infinity. For integral functions of infinite order, there seems to be no way of getting the normal distribution as a limit, because m and σ^2 are not of comparable magnitudes as $z\to\infty$. When the limit σ^2/m exists, it is the order of the integral function and the distribution for positive finite order approaches the normal under fairly general restrictions on a_n for the types of functions we use. The reason is that $f(z)$ does not then differ by much, relatively, from the greatest term in its expansion,[4] and the values of $a_n z^n/f$ taper off rapidly to zero on either side. Transferring the origin to the greatest term (relatively near the mean value, which would otherwise recede infinitely far from the origin) and changing the scale, it is easy to show, (at least when there are limited gaps in the coefficients and the coefficients themselves decrease monotonically,) that the distribution approaches a continuous one obeying the differential equation $dF/Fdx = -cx$, which characterizes our normal distributions.

Series with a unit radius of convergence might also be useful, as the Lambert series distributions again permit results of the theory of numbers to be translated bodily into statistics. But in general, it would seem impossible to get the normal distribution as a limiting case, as is shown for example by the behaviour of the Abel-summability function $f(z)=1/(1-z)$, $p_n=z^n(1-z)$, as $z\to 1$.

5. As an illustration of all the foregoing, we take a problem usually solved by other methods, and show that its fundamentals are made simpler to the understanding by use of probability. Arrange a large number of urns in serial order, labelled 1, 2, 3...n..., and let each urn contain as many balls as there are prime factors in its label-integer. Each factor is to be counted according to its multiplicity and unity may be taken as a prime only for the first urn. Now the exact probability of there being k balls in an urn chosen at random is not known, but an approximation is obtained from an asymptotic value for the number $\pi_k(x)$ of integers $\leqslant x$ containing exactly k prime factors, $\pi_k \sim \dfrac{x}{\log x}\,\dfrac{(\log\log x)^{k-1}}{(k-1)!}$

Take the value of the probability (the ratio π_k/x) obtained from this as exact. This can be transformed by taking $z=\log\log x$, $\log x = e^z$. The distribution function is seen to be ze^z; every number has at least

one prime factor, and *we have a modified Poisson distribution* as explained in section 1. The expectation of the number of balls in an urn is $\frac{\theta z e^z}{z e^z} = z + 1$. Thus we should have the total number of prime factors of all integers $\leqslant x$ as $x(\log \log x + 1)$. Now the known [6] formula for this is $x(\log \log x + B) + O(x)$. The $O(x)$ term is naturally due to the neglected remainder, as we have taken an approximation as the exact value, and summed to infinity instead of stopping at x terms of the series. The value of the constant B is a little less than unity, due not only to the two causes given above, but also to the fact that there is a linkage between the total number of urns, and the probability; the parameter z is not independent of x. As the value of B is known, we do not investigate this further.

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GLEANINGS

Die ganzen Zahlen hat Gott gemacht; Alles andere ist Menschenwerk.

KRONECKER in *Jahresbericht* Bd. 2.

A THEOREM ON CONGRUENCE

BY

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1. In this paper is discussed a generalization of the theorem that if p be an odd prime, then

$$\sum_{n=1}^{p-1} n^{p-1} - p - (p-1)! \equiv 0, \pmod{p^2}$$

which I have established elsewhere.*

1·1 We shall make use of the following notation:—

$\phi(m)$ = number of numbers less than m and prime to it. For convenience of printing it will be sometimes contracted into ϕ.

$\sum(n;\ r;\ p_1, p_2, \ldots p_n)$ = the sum of the products taken r at a time of the n numbers $p_1, p_2 \ldots p_n$;

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}; \quad \binom{\phi(m)}{r} = \phi_r$$

2. THEOREM 1. *If $n_1, n_2, \ldots n_{\phi(m)}$ be a reduced residue system, mod m, then*

$$\sum\left(\phi(m);\ r;\ n_1^{\phi}, n_2^{\phi}, \ldots n_{\phi}^{\phi}\right) + \phi_r\left(r - 1 \pm r\cdot n_1 n_2 \ldots n_{\phi}^{\phi}\right)$$

$$\equiv 0, \pmod{m^2}, \qquad \ldots \quad (1)$$

where the positive sign is to be taken when $m = 4$, p^n or $2p^n$, p being an odd prime, and the negative sign when m has any other value.

§ 2·2 The proof depends on the LEMMA: *If $p_1, p_2, \ldots p_s$ be numbers prime to m, then*

$$(p_1 p_2 \ldots p_s)^t - \sum(s;\ 1;\ p_1^t, p_2^t, \ldots p_s^t) + s - 1 \equiv 0, \pmod{m^2}, \ldots \quad (2)$$

where t is a multiple of $\phi(m)$.

This lemma is a particular case of the following

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* A Theorem on Congruence. *Jour. Indian Math. Society* Vol. II, (New series) No. 8.

THEOREM 2. *If $p_1, p_2, \cdots p_s$ be numbers prime to m, and t be a multiple of $\phi(m)$, then*

$$\begin{aligned}&(p_1 p_2 \cdots p_s)^t - \\ &\sum(s;\ r-1;\ p_1^t, p_2^t, \cdots p_s^t) + \\ &\binom{s-r+1}{1} \sum\left(s;\ r-2;\ p_1^t, \cdots p_s^t\right) - \\ &\binom{s-r+2}{2} \sum\left(s;\ r-3;\ p_1^t, \cdots p_s^t\right) + \cdots \\ &+(-)^r \binom{s-1}{r-1} \equiv 0 \pmod{m^r} \qquad \ldots \quad (3)\end{aligned}$$

Proof † $(x-p_1^t)(x-p_2^t) \cdots (x-p_s^t)$

$$\equiv (x-1)^s - S_1(x-1)^{s-1} + \cdots + (-)^{r-1} S_{r-1}(x-1)^{s-r+1} \pmod{m^r}$$

where S_q denotes $\sum(s;\ q;\ p_1^t-1,\ p_2^t-1, \cdots p_s^t-1)$
since $S_r, S_{r+1}, \cdots S_s$ are $\equiv 0 \pmod{m^r}$, by Euler's generalization of Fermat's theorem.

Putting $x=0$ in (4) we get

$$p_1^t\, p_2^t \cdots p_s^t \equiv (1 + S_1 + S_2 + \cdots + S_{r-1}) \pmod{m^r} \qquad \ldots \quad (5)$$

Now $1 + S_1 + S_2 + \cdots + S_{r-1} =$ coefficient x^{r-1} in the expansion in powers of x of $(1-x+xp_1^t)(1-x+xp_2^t) \cdots (1-x+xp_s^t)(1-x)^{-1}$

$=$ coefficient of x^{r-1} in

$$(1-x)^{s-1} + x(1-x)^{s-2} \sum(s;\ 1;\ p_1^t, \cdots p_s^t) + \cdots\cdot$$

$$= (-)^{r-1}\binom{s-1}{r-1} + (-)^{r-2}\binom{s-2}{r-2} \sum\left(s;\ 1;\ p_1^t, \cdots p_s^t) + \cdots\cdot\right)$$

Substituting this value in (5) we get the required result.

3. *Proof of Theorem* 1.

In (2) take the p's to be the products, r at a time, of the numbers $n_1, n_2, \ldots n_{\phi(m)}$.

Then we get

$$\left(n_1, n_2 \ldots n_{\phi(m)}\right)^{r\phi_r} - \sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_{\phi}^{\phi}\right) + \binom{\phi(m)}{r} - 1 \equiv 0, \pmod{m^2}.$$

† The theorem can also be proved by induction on s.

By Gauss's generalization of Wilson's theorem we have,

$$n_1\, n_2 \ldots n_{\phi(m)} \equiv \begin{cases} -1;\; m=4,\ p^n,\ 2p^n \\ +1;\; m \neq 4,\ p^n,\ 2p^n \end{cases};\ p \text{ being an odd prime.}$$

Let $n_1\, n_2 \ldots n_{\phi(m)} = \lambda m \mp 1$.

Then $$\left(\lambda m \mp 1\right)^{r\phi_r} - \sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_\phi^{\phi}\right) + \left(\phi_r(m)\right) - 1 \equiv 0,\ (\text{mod } m^2)$$

Therefore, noting that $\phi(m)$ is even if $m>2$, we get

$$1 \mp \lambda m^{\phi_r}\, r - \sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_\phi^{\phi}\right) + \phi_r - 1 \equiv 0,\ (\text{mod } m^2).$$

i.e. $$\sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_\phi^{\phi}\right) + \phi_r\left[r - 1 \pm r \cdot n_1\, n_2 \ldots n_{\phi(m)}\right] \equiv 0,\ (\text{mod } m^2).$$

COROLLARY

Putting $r=1$, we get

$$\sum_{r=1}^{\phi(m)} n_r^{\phi(m)} \pm \phi(m)\, n_1\, n_2 \cdots n_\phi \equiv 0\ (\text{mod } m^2)$$

THEOREM 3. *If x be prime to m,*

$$x^{s\phi(m)} - \binom{s}{r-1} x^{(r-1)\phi(m)} + \binom{s}{r-2}\binom{s-r+1}{1} x^{(r-2)\phi(m)} - \ldots + (-)^r \binom{s-1}{r-1} \equiv 0,\ (\text{mod } m^r)$$

This is easily proved by putting $p_1 = p_2 = \cdots = p_s = x,\ t = \phi(m)$ in theorem 2.

GLEANINGS

SOUND

The Soul, desirous of speaking out its intention, excites the Mind and the Mind operates on the Vital Heat of the body by setting the Air in motion, and the Air remaining in the *Brahmagranthi* rises up and produces Sound, through the navel, the heart, the neck, the head and the face.

RAMAMATYA in *Svaramelakalanidhi*, (16th century).

THE ROLE OF UNIT PARTIAL QUOTIENTS IN SOME CONTINUED FRACTIONS

BY

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1. The unit partial fractions, if any, in a simple continued fraction (briefly S. C. F.) play a significant role. They form the link between the simple continued fraction and the several half regular continued fractions (briefly *h. r.* C. F's). By their annihilation†[1] we obtain the most rapidly convergent *h. r.* C. F's as well as those of shortest length when the C. F's are finite. Minkowsky's[2] diagonal C. F. development of a real number other than an integer or half an integer is also based on the annihilation, under certain circumstances, of special unit partial quotients, viz, those that immediately follow or precede a partial quotient greater than 1. In this connection, we may point out a property related to diagonal C. F's not probably noticed before:

'*A unit partial quotient is no longer 'special' or 'ausgezeichnete' when it is preceded and followed by a unit partial quotient.*'††

This shows that the first and the last in a set of successive unit partial quotients (in between two partial quotients greater than 1) have a special character distinct from that of the intervening ones. We shall see presently another context in which a similar phenomenon occurs. We propose to discuss the properties of a class of pure recurring S. C. F's with special reference to the unit partial quotients.

For convenience, the following conventions will be adopted:

$$(1)\quad \xi_0=(a_0,\ a_1,\ a_2,\ .\ .\ a_v,\ .\ .\ .\ )=a_0+\frac{1}{a_1}+\frac{1}{a_2}+\ .\ .\ .\ \frac{1}{a_v}+\cdots;$$

ξ_v is the $(v+1)$-th complete quotient and corresponds to the partial quotient a_v.

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† This and other subsequent numbers in brackets appearing between consecutive lines relate to references at the end of the paper.

†† On p. 184 footnote, Perron[3] has pointed out that $b_{\lambda+1}=1$ $(\lambda>0)$ is a special partial quotient if $(0,\ 1+b_{\lambda+3},\ b_{\lambda+4},\ldots)\geqslant(0,\ b_\lambda,\ b_{\lambda-1},\ldots\ b_1)$, when $b_{\lambda+2}=1$. We now remark that this inequality is not satisfied, if $b_\lambda=1$.

(2) $\overline{(a_0, a_1, a_2, \cdots a_r, 1_{[n]}, a_{r+n+1}, \cdots a_p)}$

$$=a_0+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_r}+\frac{1}{1}+\frac{1}{1}+\cdots\frac{1}{1}+\frac{1}{a_{r+n+1}}+\cdots\frac{1}{a_p}$$

(a pure recurring S. C. F with n unit partial quotients between a_r and a_{r+n+1});

(3) A quadratic surd of the form $(P+\sqrt{R})/Q$ is said to be standard, when P, Q, $(R-P^2)/Q$ are integers having no common factor R being a non-square positive integer.

2. THEOREM I:—*If* $\xi_0=(P_0+\sqrt{R})/Q_0$ *a standard surd develops as a pure recurring simple continued fraction with a set of successive unit partial quotients preceded and followed by other partial quotients, then the denominators of the complete quotients corresponding to the unit partial quotients other than the first and the last of the set are less than* $\sqrt{R}$.

Proof:—Let $\xi_0=\dfrac{P_0+\sqrt{R}}{Q_0}=\overline{(a_0, a_1, \cdots a_r, 1_{[n]}, a_{r+n+1}, \cdots a_p)}$.

Then $\xi_{r+v}=\dfrac{P_{r+v}+\sqrt{R}}{Q_{r+v}}=\overline{(1_{[n-v+1]}, a_{r+n+1}, \cdots a_p, a_0, a_1, \cdots a_r, 1_{[v-1]})}$

$= f$ (say) when $1 \leqslant v \leqslant n$ ... (2·1)

By Galois' theorem[4] of inverse periods,

$$\frac{Q_{r+v}}{\sqrt{R}-P_{r+v}}=\overline{(1_{[v-1]}, a_r, \cdots a_1, a_0, a_p, \cdots a_{r+n+1}, 1_{[n-v+1]})}$$

i.e. $\dfrac{\sqrt{R}-P_{r+v}}{Q_{r+v}}=(0, \overline{1_{[v-1]}, a_r, \cdots 1_{[n-v+1]}})=f'$ (say) ... (2·2)

Adding (2·1) and (2·2), $\dfrac{2\sqrt{R}}{Q_{r+v}}=f+f'$, so that Q_{r+v} is greater or less than $\sqrt{R}$ according as f is less or greater than $(2-f')$ or f'' (say).

Now, $2-f' = 2-(0, 1_{[v-1]}, \cdots)$
$=(1, 1, 0, 1_{[v-2]}, \cdots)$
$=(1, 2, 1_{[v-3]}, \cdots)$

If $v \geqslant 3$ and $(n-v+1) \geqslant 2$, i.e. $n-1 \geqslant v \geqslant 3$, the second complete quotient of f, is less than the corresponding quotient of f'' and therefore $f > f''$ so that $Q_{r+v} < \sqrt{R}$.

If $v=2$, and $n \geqslant 3$, we have $f''=(1, 1+a_r, \cdots) < f$ and again $Q_{r+v} < \sqrt{R}$.

Thus, for all values of v greater than 1 and less than $n (\geqslant 3)$, $Q_{r+v} < \sqrt{R}$.

The theorem is therefore proved.

Cor. (1) When $v=1$, $n>2$ and $a_r>2$, we have $f<f'$ and therefore $Q_{r+1}>\sqrt{R}$.

If $v=1$, $n=2$, we find Q_{r+1} greater or less than $\sqrt{R}$ according as a_{r+3} is less or greater than a_r-1; if $a_r=2$, $Q_{r+1}<\sqrt{R}$.

If $n=v=1$, $f<f''$ and $Q_{r+1}>\sqrt{R}$.

Cor. (2) When $v=n>2$, we can show that $Q_{r+n}>\sqrt{R}$, if $a_{r+n+1}>2$.

When $v=n=2$, Q_{r+2} is greater or less than $\sqrt{R}$ according as a_{r+3} is greater or less than a_r+1: if $a_{r+3}=2$, $Q_{r+2}<\sqrt{R}$.

If $n=2$ and $a_{r+3}=a_r$, both Q_{r+1} and Q_{r+2} are less than $\sqrt{R}$.

N. B. The complete quotients corresponding to the first and the last of a set of unit partial quotients preceded and followed by partial quotients greater than 1, may have their denominators greater or less than $\sqrt{R}$ according to circumstances and are thus distinguished from the other complete quotients whose denominators are always less than $\sqrt{R}$. From the[5] theory of continued fractions to the nearest square (or otherwise) it will be seen that in a pure recurring simple continued fraction equivalent to $\frac{P_0+\sqrt{R}}{Q_0}$, the denominators of the complete quotients are all less than $\sqrt{R}$, when the corresponding partial quotients are greater than 1.

3. THEOREM II:—*In the S. C. F development of a standard surd of the form* $\frac{q+\sqrt{p^2+q^2}}{p}$, $(p>2q>0)$ *there cannot occur a complete quotient of the same form more than once, in the recurring period; when such a complete quotient does occur, the recurring period is symmetric with an even number of terms, which include in the middle an even number of unit partial quotients.*

$$\textit{Proof:}\text{—Let } \xi_0=\frac{P_0+\sqrt{R}}{Q_0}=\frac{q+\sqrt{p^2+q^2}}{p}, \qquad [p>2q>0.]$$

Then it is readily seen that
$$1<\xi_0<\frac{\sqrt{5}+1}{2}=(1\infty) \qquad \ldots \quad (3\cdot 1)$$

while $\bar{\xi}_0$ (the conjugate of ξ_0) $=\frac{q-\sqrt{p^2+q^2}}{p}$, so that

$$\xi_0\,\bar{\xi}_0=-1 \text{ and } -1<\bar{\xi}_0<0.$$

Hence, by a well known theorem[6] of Galois, the S. C. F. development of ξ_0 has a pure recurring period, say $(\overline{a_0, a_1, \cdots a_n})$.

From (1), $a_0 = 1$ and if a_m be the first partial quotient greater than 1, m must be odd; for, if m be even, we have successively:

$$(a_m, \cdots\cdots) > (1\infty),$$
$$(1, a_m, \cdots\cdots) < (1\infty),$$
$$(1[2], a_m, \cdots\cdots) > (1\infty),$$
$$\cdots\cdots\cdots$$
$$(1[m], a_m, \cdots) > (1\infty), \text{ which contradicts } (3{\cdot}1)$$

Hence $$\xi_0 = (\overline{1[m], a_m, \cdots a_n}). \qquad \ldots \quad (3{\cdot}2)$$

Again, from the theorem of inverse periods,

$$\xi_0 = -\frac{1}{\bar{\xi}_0} = (\overline{a_n, \cdots 1_{[m]}}) \qquad \ldots \quad (3{\cdot}3)$$

Comparing (3·2) and (3·3), we have

$$a_n = a_{n-1} = \cdots = a_{n-m+1} = 1, \ a_m = a_{n-m};$$

and the period is a symmetric one beginning and ending with an odd number of unit partial quotients.

The comparison of the complete quotients in (3·2) and (3·3) gives, when $v \leqslant n$,

$$\frac{P_v + \sqrt{R}}{Q_v} = \frac{P_{n+1-v} + \sqrt{R}}{Q_{n-v}}, \text{ so that } P_v = P_{n-v+1}, \ Q_v = Q_{n-v} \ \ldots \quad (3{\cdot}4)$$

If $Q_v = Q_{v-1}$, then $\xi_v = \dfrac{P_v + \sqrt{R}}{Q_v} = \dfrac{P_{n-v+1} + \sqrt{R}}{Q_{v-1}} = \xi_{n-v+1}$

since $Q_{v-1} = Q_{n-v+1}$ by (3·4).

Hence, $v = n + 1 - v$, i.e. $v = \dfrac{n+1}{2}$.

This shows that, if two consecutive Q's should be equal, n must be odd and the equal Q's are $Q_{\frac{n+1}{2}}$ and $Q_{\frac{n-1}{2}}$ and no others. ... (3·5)

If a complete quotient, say ξ_v, of the recurring period of ξ_0 be of the same form as ξ_0, its S. C. F development should possess the same symmetries.

Writing $Q_0, Q_1, \cdots Q_n$ round a circle at the vertices of a regular polygon of $(n+1)$ sides, we find that they arrange themselves symmetrically about a diameter such that the Q's symmetrically placed about this diameter are also equal, since $Q_v = Q_{n-v}$.

The symmetry of the Q's corresponding to ξ_v imply that $Q_v = Q_{v-1}$ just as $Q_0 = Q_n$.

From (3·5), we see that there cannot be more than one ξ_v of the same form as ξ_0 within the recurring period and it can occur only when n is odd and $v = \frac{1}{2}(n+1)$.

Further, when n is odd, we realise the same symmetry of Q's starting from $Q_{\frac{n+1}{2}}$, going round the circle and ending with $Q_{\frac{n-1}{2}}$ as in the first set $(Q_0, Q_1, \cdots Q_n)$.

This proves the existence of ξ_v of the same form as ξ_0, only when $a_{(n \pm r)/2} = 1$, where $r = 1, 3, 5, \cdots (2m-1)$, and m, n are both odd.

Hence, if ξ_0 should have a remote successor of the same form as itself in the recurring period of its S. C. F development,

then $\xi_0 = (\overline{1_{[2k+1]}, a_{2k+1}, \cdots a_l, 1_{[4k'+2]}, a_l, \cdots a_{2k+1}, 1_{[2k+1]}})$ with a symmetric set of partial quotients, where a_{2k+1}, a_l are not equal to 1.

In order that the recurring cycle may preserve its character as a primitive period, it is necessary that

$a_{2k+1} \neq a_l$, $k \neq k'$, and $a_p \neq a_q$ where $(2k+1) < (p, q) < l$, the number of terms in the cycle being $2p+2q+2$.

Example: $\dfrac{27+\sqrt{27^2+82^2}}{82} = (\overline{1, 2, 1_{[6]}, 2, 1})$ has a remote successor

$$\frac{37+\sqrt{37^2+78^2}}{78} = (\overline{1_{[3]}, 2, 1_{[2]}, 2, 1_{[3]}})$$

4. We will next proceed to examine, under what circumstances a standard quadratic surd of the form $\sqrt{R}/Q_0$ can have a complete quotient of the form in Theorem II in its S. C. F development, say

$$(a_0, \overline{a_1, a_2, \cdots a_{k-1}, 2a_0}) \qquad \ldots \text{(A)}$$

Since the first and the last partial quotients in the period of $\dfrac{q+\sqrt{p^2+q^2}}{p}$, $(p > 2q > 0)$, are unity, while $2a_0$ can never be equal to 1, a complete quotient, say ξ_v, if any, of the form in question occurring in the recurring period of (A), cannot correspond to either a_1 or $2a_0$, so that

$$\xi_v = (\overline{a_v, \cdots a_{v-1}}), \; a_v \neq a_1 \text{ or } 2a_0.$$

From the known property of the Q's, we have $Q_v = Q_{v-1}$, which implies by a well-known theorem[6] of Muir, that k must be odd, and in this case $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, where $R = P_v^2 + Q_v^2$.

Further, there cannot be another complete quotient of the same form in the recurring period, which is possible only when k is even (vide (3·5) supra).

From the symmetries in the recurring period in (A), we infer that

$$\xi_{\frac{k+1}{2}} = (\overline{a_{\frac{k+1}{2}}, \cdot \cdot a_{k-1}, 2a_0, a_1, \cdot \cdot a_{\frac{k-1}{2}}}).$$

Thus $\sqrt{R}/Q_0$ must contain in its S. C. F period an even number $(4n+2)$ of unit partial quotients in the centre of the symmetric part of the period in order that one of its complete quotients may be of the form $\frac{q+\sqrt{p^2+q^2}}{p}$ $(p > 2q > 0)$.

Examples: $\sqrt{58} = (7, \overline{1[6], 14})$; $\sqrt{97} = (9, \overline{1, 5, 1[6], 5, 1, 18})$
We condense the above results in the following

THEOREM III :—*If the standard surd of the form $(\sqrt{R})/Q_0$ have in its S. C. F development a complete quotient of the form* $\frac{q+\sqrt{p^2+q^2}}{p}$, $p > 2q > 0$, *then the symmetric portion of the recurring period of partial quotients will contain in the middle an even number (of the form $4n+2$) of unit partial quotients; and there cannot occur any other complete quotient of the same form within the recurring period.*

5. Before concluding, we wish to point out an application of the above theorems in connection with the most rapidly convergent half regular continued fractions.

According to[7] Tietze's theorem, the most rapidly convergent *h. r. c.* F's are characterised by the property that the complete quotients are everywhere greater than $\frac{1}{2}(\sqrt{5}+1)$. We have proved elsewhere that the only transformations (apart from the P-transformation) that go to convert a S. C. F into one of the most rapidly convergent *h. r.* C. F's are the annihilatory transformations which we have called the C_1, C_2, and $c_1\ c_2^e$ types. The effect of an annihilatory transformation applied to a unit partial quotient is obviously to increase the following complete quotient by 1, without affecting the preceding complete quotient.

From the above considerations, we see that a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$ will occur in one of the most rapidly convergent *h. r.* C. F developments of $(\sqrt{R})/Q_0$ (not involving a P-transformation), when and only when either $\frac{q+\sqrt{p^2+q^2}}{p}$ or $\frac{q+p+\sqrt{p^2+q^2}}{p}$ occurs in the corresponding S. C. F development.

Since $\frac{q+p+\sqrt{p^2+q^2}}{p}$ is not a reduced surd in[3] Perron's sense and cannot therefore occur in the S. C. F recurring period, while $\frac{q+\sqrt{p^2+q^2}}{p}$ can occur just once in the same period under the conditions of Theorem III, it follows that a most rapidly convergent *h. r.* C. F development of $(\sqrt{R})/Q_0$ (not involving a P-transformation) will have in its period $\frac{q+p+\sqrt{p^2+q^2}}{p}$ as a complete quotient just once, when the unit partial quotient corresponding to $\frac{q+\sqrt{p^2+q^2}}{p}$ in the S. C. F is not annihilated.

Let $(\sqrt{R})/Q_0 = (a_0, \overline{a_1, \cdot \cdot a_p, \dot{1}_{[4t+2]}, a_p, \cdot \cdot \cdot a_1, 2a_0})$, where ξ_{p+2t+3} is the only complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$. The result of applying the C_1-transformation gives the complete quotient $1+\xi_{p+2t+3}$; if the C_2-transformation be applied, the unit partial quotient corresponding to ξ_{p+2t+3} will be annihilated and so there will be no complete quotient of the form in question. We have, therefore,

THEOREM IV:—*In the development of $(\sqrt{R})/Q_0$ as a continued fraction to the nearest integer, there cannot occur more than one complete quotient of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}$ in the recurring period, while in the singular continued fraction development, such a complete quotient can never occur.*

To preserve the complete quotient $1+\xi_{p+2t+3}$, we may also apply the eclectic transformation $c_1\ c_2^e$, provided the C_1-process is continued at least until it annihilates the $(2t+1)$-th unit partial quotient.

Lastly, we may state that the C_1-transformation applied to odd batches of unit partial quotients and the $c_1\ c_2^e$ transformation to even batches of a pure recurring S. C. F. (ξ_0), will remove the special unit partial quotients and then, by Theorem I, the denominators of all complete quotients are necessarily less than $\sqrt{R}$.

The results set down in this paper are useful in the theory of the continued fraction to the nearest square.[8]

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(7) Über die raschesten Kettenbruchentwicklungen reeller Zahlen, by H. Tietze (*Monatshefte für Mathematik und Physik*, Bd. 24, 1913.)

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† This issue will be published shortly.

GLEANINGS

The same channels and the same relations which led the Arabs to a knowledge of Indian Algebra, enabled them also to obtain, in the ninth century, Indian numerals from Persia and the shores of Euphrates. Persians were established at that time as Revenue Collectors on the Indus, and the use of Indian numerals was gradually transmitted to the Revenue Officers of the Arabs in Northern Africa, opposite the shores of Sicily. Nevertheless, the important historical investigations of the distinguished mathematician Charles have rendered it more than probable according to his correct interpretation of the so-called Pythagorean table in the geometry of Bœthius, that the Christians in the west were familiar with Indian numerals even earlier than the Arabs, and that they were acquainted with the use of nine figures or characters according to their position value under the name of the system of the abacus.

HUMBOLDT,

Quoted from *Fabre and Mathematics and other essays*, p. 39

THE INFINITESIMAL FLOW IN THE INVERSIVE PLANE

BY

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Given any continuous r-parameter group G_r of point transformations we can associate with it a system of r infinitesimal operators

$$X_r = \sum \xi_r \frac{\partial}{\partial x_r}$$

such that any infinitesimal transformation can be expressed in the form

$$x_i' = x_i + \sum_a c^a \xi_a^i \, \delta t$$

where δt is a scalar infinitesimal. Interpreting δt as a time interval these operators determine a velocity field **v** which is charateristic of this group. I study here the velocity field **v** due to the inversive group.

The Infinitesimal operators of the Inversive group:

Topologically the real inversive plane is homeomorphic with the 2 sphere, the correspondence between points being most easily established by streographic projection. Under such a correspondence the operators of the inversive group in the plane correspond to projective transformations which transforms the quadric into itself. To obtain the infinitesimal operators, we notice that $\lambda = x + iy$ and $\mu = x - iy$ may be taken as the "generator coordinates" on the sphere and correspond to the 2 sets of isotropic lines in the cartesian plane. A projective transformation of the sphere which does not interchange the 2 systems of generators correspond to the linear group $\lambda' = (a_1\lambda + b_1)/(c_1\lambda + d_1)$ on the one system together with a transformation $\mu' = (a_2\mu + b_2)/(c_2\mu + d_2)$ of μ generators. The infinitesimal operators in the λ and μ coordinates are thus:

$$\frac{\partial}{\partial \lambda},\ \lambda \frac{\partial}{\partial \lambda},\ \lambda^2 \frac{\partial}{\partial \lambda},\ \frac{\partial}{\partial \mu},\ \mu \frac{\partial}{\partial \mu},\ \mu^2 \frac{\partial}{\partial \mu}. \quad \ldots \quad (1)$$

From these we derive by substituting in terms of x and y on forming real combinations the infinitesimal operators of the inversive group of the plane, namely:

$$\text{(I)}\ \frac{\partial}{\partial x}, \qquad \text{(II)}\ \frac{\partial}{\partial y};$$

$$\text{(III)} \quad x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x};$$

$$\text{(IV)} \quad x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y};$$

$$\text{(V)} \quad 2xy \frac{\partial}{\partial x} + (y^2 - x^2) \frac{\partial}{\partial y},$$

$$\text{(VI)} \quad 2xy \frac{\partial}{\partial y} + (x^2 - y^2) \frac{\partial}{\partial x}. \qquad \ldots \quad (2)$$

Of these, the first two operators represent small displacements parallel to the x and y axes respectively; the third represents a small rotation about the origin; the fourth represents a small uniform expansion with the origin as a fixed point. The first three of these generate the *metrical group*, while if the fourth is also adjoined we have the *affine group*. The projective group of the plane is obtained by adjoining to the operators, (I), (II), (III), and (IV) the operators

$$x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}, \quad x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.$$

The operators V and VI are thus peculiar to the inversive group. The fifth corresponds to an inversion with respect to a circle of unit radius whose centre is at the origin followed by an inversion with respect to a circle of unit radius whose centre is at a small distance δt from the origin on the y axis followed by a translation back-wards along the axis of y measured by δt. The operator VI has a similar interpretation in relation to the axis of x.

Other subgroups that can be generated from these operators are:—

(*a*) The group generated by the operators III, IV, V and VI which keeps the origin invariant and (*b*) groups generated by the operators I, IV and VI. and II, IV, V; the group generated by I, IV and VI keeps the x axis invariant and the other keeps the y axis invariant.

The Components of the Flow Vector. Any general infinitesimal operator X of the Inversive group of the plane G_6 is linearly dependent upon the above six operators. Hence it may be represented by

$$X \equiv a_0 \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial y} + a_2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + a_3 \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$
$$+ a_4 \left\{ 2xy \frac{\partial}{\partial x} + (y^2 - x^2) \frac{\partial}{\partial y} \right\} + a_5 \left\{ 2xy \frac{\partial}{\partial y} + (x^2 - y^2) \frac{\partial}{\partial x} \right\}$$

$$\equiv \left\{ a_0 - a_2\, y + a_3\, x + 2a_4\, xy + a_5\, (x^2 - y^2) \right\} \frac{\partial}{\partial x}$$

$$+ \left\{ a_1 + a_2\, x + a_3\, y + a_4\, (y^2 - x^2) + 2a_5\, xy \right\} \frac{\partial}{\partial y} \qquad \ldots \quad (3)$$

where a_0, a_1, a_2, a_3, a_4, a_5, are constants.

The velocity vector $\mathbf{v}$ is given by

$$\mathbf{v} = \mathbf{i}\ v_0 + \mathbf{j}\ v_1 \qquad \ldots \quad (4)$$

where $v_0 = \{a_0 - a_2\, y + a_3\, x + 2a_4\, xy + a_5\, (x^2 - y^2)\}$

$v_1 = \{a_1 + a_2\, x + a_3\, y + a_4\, (y^2 - x^2) + 2a_5\, xy\}$.

Calculating div $\mathbf{v}$ and curl $\mathbf{v}$ from (3) we have:—

$$\text{div } \mathbf{v} = \frac{\partial v_0}{\partial x} + \frac{\partial v_1}{\partial y}$$

$$= 2\ (a_3 + 2a_4\, y + 2a_5\, x) \qquad .. \quad (5)$$

and

$$\text{curl } \mathbf{v} \equiv \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_0}{\partial y} \right) \mathbf{k}.$$

$$= 2\ (a_2 - 2a_4\, x + 2a_5\, y)\ \mathbf{k}. \qquad .. \quad (6)$$

where $\mathbf{k}$ is a unit vector normal to the plane.

From equations (5) and (6) we see that the loci of points at which div $\mathbf{v}$ and curl $\mathbf{v}$ are constants are given by two systems of lines which cut orthogonally.

Taking the lines along which div $\mathbf{v}$ and curl $\mathbf{v}$ vanish as the axes of x and y respectively we have

$$a_3 + 2a_4\, y + 2a_5\, x \equiv y' \equiv 0, \text{ i.e. } a_3 \equiv a_5 \equiv 0 \qquad \ldots \quad (7)$$

and

$$a_2 - 2a_4\, x + 2a_5\, y \equiv x' \equiv 0, \text{ i.e. } a_2 \equiv a_5 \equiv 0 \qquad \ldots \quad (8)$$

Hence by choosing this set of new axes we have the new velocity vector $\mathbf{v}'$ given by the equation $\mathbf{v}' = \mathbf{i}v_0' + \mathbf{j}v_1'$ ... (9)

where $v_0' = a_0 + 2a_4\, xy$

$v_1' = a_1 + a_4\, (x^2 - y^2)$.

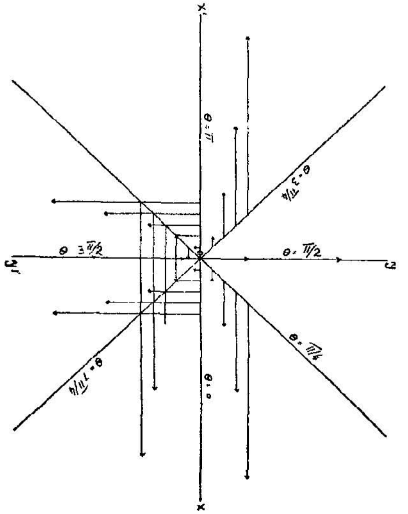
This velocity vector $\mathbf{v}'$ can be separated into two velocity vectors :—

(*a*) $\mathbf{v}_1 = \mathbf{i}a_0 + \mathbf{j}a_1$ due to a pure translation:

and (*b*) $\mathbf{v}_2 = 2a_4\, xy\, \mathbf{i} + a_4\, (y^2 - x^2)\ \mathbf{j}$.

which is peculiar to the inversive group.

The velocity vector corresponding to $\mathbf{v}_2$ is obtained as follows :— changing to polar coordinates we see that the displacements along the x and y axes are given by $r^2 \cos(2\theta - 90°)$ and $r^2 \sin(2\theta - 90°)$.



From the above formula it may be verified that the velocity vector at all points on a straight line through the origin are parallel and similarly directed while the magnitude varies as the square of the distance of the point from the origin. A fairly good idea of the direction and magnitude of the velocity vectors at different points in the plane, may be had from fig. 1.

The direction of the velocity vector can also be obtained as follows: Suppose the circle of unit radius with centre at O′ rolls without slipping on another unit circle with centre O when the point of contact is on the x axis (the line of centres being the x axis). Let AB be the diameter parallel to the y axis through O′. Then the direction of AB gives the direction of the velocity vector of O′.

Before concluding I wish to express my thanks to Professor A. Narasinga Rao for suggesting this problem and for his guidance in this investigation.

GLEANINGS

The geometry of your face,
The algebra of your eyes—with their knowns and unknowns,
The calculus of your personality—with its constants and variables,
The radii of your charm that encircles me,
Your very locus,
The force of your smile, that produced becomes the perpendicular bisector of my heart,
And leaves me entangled in the graph you have plotted around me :—

CATHERINE M. WILLIMEN,
(*From The Mathematics Teacher, Feb. 1941*)

THE LINE OF UMBILICS

BY

DR. C. N. SRINIVASIENGAR, *Bangalore.*

1. At an umbilic on a surface, it is well-known that there are three principal directions in the sense that the normal at the umbilic meets the normal at a consecutive point. The object of this note is to point out the following result, which does not appear to have been noticed, as far as I am aware.

If a surface possesses a locus of umbilics, the tangent to this locus at any point is one of the three principal directions thereat. In other words, a line of umbilics on a surface is a line of curvature.

Taking the origin at an umbilic, the equation of the surface may be written in the form:

$$2z = \frac{x^2+y^2}{\rho} + \tfrac{1}{3}\,(ax^3 + 3bx^2y + 3cxy^2 + dy^3) + \cdots$$

If $(x, y, 0)$ be a neighbouring umbilic, x and y being small, we obtain from the differential equations

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t},$$

$$\frac{1+\dfrac{x^2}{\rho^2}+\cdots}{\dfrac{1}{\rho}+ax+by+\cdots} = \frac{xy/\rho^2}{bx+cy+\cdots} = \frac{1+\dfrac{y^2}{\rho^2}}{\dfrac{1}{\rho}+cx+dy+\cdots}$$

It follows that the denominator of the second fraction should be an infinitesimal of the second order. Hence, to the first order of infinitesimals, we must have

$$bx + cy = 0$$

$$\text{and } (a-c)\,x + (b-d)\,y = 0.$$

Hence,
$$b\,(b-d) = c\,(a-c). \tag{1}$$

Now, the three principal directions at O are given by*

$$c\tan^3\alpha + (2b-d)\tan^2\alpha - (2c-a)\tan\alpha - b = 0 \tag{2}$$

* Bell. *Coordinate Geometry of Three Dimensions*, p. 343.

where α is the angle that any of the directions makes with the x-axis. It is seen that $\tan \alpha = y/x = -b/c$ satisfies the equation (2) in virtue of the relation (1). This proves the result.

2. From Rodrigue's formulae for a line of curvature, we have

$$\frac{dx}{dX} = \frac{dy}{dY} = \frac{dz}{dZ}$$

where X, Y, Z are the direction cosines of the normal at (x, y, z). It is now easily seen that these equations are satisfied also, by a line of umbilics if such a line be present on any surface. This fact has a bearing on the following theorem of Joachimsthal namely:

If two surfaces intersect at a constant angle, and if the curve of intersection is a line of curvature upon one of the surfaces, it is a line of curvature on the other also.

Two proofs* are known for this theorem, one by using Rodrigue's formulae, the other by using the formulae for geodesic torsion. The second method will explain that the curve of intersection can also possibly be a line of umbilics on the second surface. The property proved in §1 of this note serves to explain that Joachimsthal's theorem remains *literally* true even in this exceptional case. That the first method of proof also furnishes this alternative is now clear from the remark made above regarding Rodrigue's formulae.

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* Weatherburn, *Differential Geometry* I, Arts. 29 and 49.
Forsyth, *Differential Geometry*, Arts. 107 and 128.

NOTES AND DISCUSSIONS

A Proof of an elementary theorem on complex numbers.

If two complex numbers w_1, w_2 are derived from two complex numbers z_1, z_2 by the pair of formulae

$$w_1 = az_1 + bz_2, \qquad w_2 = cz_1 + dz_2,$$

in which the coefficients a, b, c, d are real, then rotation through an angle less than two right angles from w_1 to w_2 is in the same direction as rotation through an angle less than two right angles from z_1 to z_2 or is in the opposite direction according as $ad - bc$ is positive or negative.

This theorem is familiar, but it is usually proved by dissection of the complex numbers into real and imaginary parts, and a proof in which there is no dissection is interesting.

The coefficients c, d are not both zero, and we suppose c not to be zero. Then writing Z, W for z_1/z_2, w_1/w_2 we have

$$\mathrm{W} = \frac{a\mathrm{Z} + b}{c\mathrm{Z} + d},$$

that is,

$$\left(\mathrm{W} - \frac{a}{c}\right)\left(\mathrm{Z} + \frac{d}{c}\right) = \frac{bc - ad}{c^2}.$$

Since the product of the two complex numbers

$$\mathrm{W} - \frac{a}{c}, \qquad \mathrm{Z} + \frac{d}{c}$$

is real, the first of these complex numbers is a real multiple of the conjugate of the second. If the multiplying factor is positive, then (i) the imaginary parts of the two complex numbers have opposite signs, (ii) the product of the two complex numbers is positive. On the other hand, if the multiplying factor is negative, then (i) the imaginary parts of the two complex numbers have the same sign, (ii) the product of the two complex numbers is negative. But (i) since a/c, d/c are real, the imaginary parts of the two complex factors on the left-hand side of the last equation are simply the imaginary parts of W and Z themselves; (ii) since c^2 is essentially positive, the product is positive or negative according as $ad - bc$ is negative or positive. Thus the imaginary parts of W and Z have opposite signs or the same sign according as $ad - bc$ is negative or positive, and this is the result which was to be established.

Sonning on Thames, England. — E. H. NEVILLE.

Power Series for Circular Functions.

1. The juvenile enthusiast in Trigonometry who has mastered the elegant expansions in powers of x for sin x and cos x is apt to be disappointed when he is abruptly told, as in Loney's *Plane Trigonometry* (Part II), article 34, that 'there is no series, proceeding according to a simple law, for the expansion of tan x in powers of x, and though we can find the series for tan x to as many places as we please, the method however soon becomes very cumbrous and troublesome'. In what follows assuming the existence of the series in question, I obtain the coefficients in the power series for tan x and cot x, cosec x and sec x in the form of elegant determinants.

2. Since tan x is an odd function of x we assume

$$\tan x = T_1 x + T_3 x^3 + T_5 x^5 + \cdots + T_{2k-1} x^{2k-1} + \cdots \quad \ldots \quad (1)$$

Then the identity $\sin x = \cos x \times \tan x$ gives

$$\left\{ x - \frac{x^3}{3!} + \cdots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + \cdots \right\}$$

$$= \left\{ 1 - \frac{x^2}{2!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \right\}$$

$$\times \left\{ T_1 x + T_3 x^3 + \cdots + T_{2k-1} x^{2k-1} + \cdots \right\} \quad \ldots \quad (2)$$

whence, equating the coefficients of x^{2k-1} on both sides, we have

$$\frac{1}{(2k-1)!} - \frac{T_1}{(2k-2)!} + \frac{T_3}{(2k-4)!} - \cdots + (-1)^{k-1} \frac{T_{2k-3}}{2!} + (-1)^k T_{2k-1} = 0 \quad (3)$$

Putting in (3) $k = 1, 2, 3, \cdots k$ in succession, we have

$$\frac{1}{1!} - T_1 = 0$$

$$\frac{1}{3!} - \frac{T_1}{2!} + T_3 = 0$$

$$\frac{1}{5!} - \frac{T_1}{4!} + \frac{T_3}{2!} - T_5 = 0 \quad \ldots \quad (4)$$

. . . .

$$\frac{1}{(2k-1)!} - \frac{T_1}{(2k-2)!} + \frac{T_3}{(2k-4)!} \cdots + (-1)^{k-1} \frac{T_{2k-3}}{2!} + (-1)^k T_{2k-1} = 0$$

From these k equations eliminating the $(k-1)$ quantities

$-T_1, T_3, -T_5, \cdots (-1)^{k-1} T_{2k-3}$, we obtain

$$T_{2k-1} = \begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 & \cdot & \cdot & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdot & \cdot & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(2k-1)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdot & \cdot & \cdot & \frac{1}{2!} \end{vmatrix} = \Delta(k), \text{ say, } \ldots \quad (5)$$

Setting $k=1, 2, 3, \cdots$ in (5) it is easily calculated that

$T_1 = 1$, $T_3 = \frac{1}{3}$, $T_5 = \frac{2}{15}$, $T_7 = \frac{17}{315}$, $\cdots$ Thus we have

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17}{315} x^7 + \cdots + \Delta(k)\, x^{2k-1} + \cdots \quad \ldots (6)$$

3. Similarly, we observe that $x \cot x$ is an even function, such that $\lim_{x \to 0} (x \cot x) = 1$. Hence we write

$$x \cot x = 1 + C_2 x^2 + C_4 x^4 + \cdots + C_{2k} x^{2k} + \cdots \quad \ldots (7)$$

The identity $\cos x = \left(\frac{\sin x}{x}\right)(x \cot x)$ then gives on equating the coefficient of x^{2k} on both sides, after a slight reduction,

$$\frac{2k}{(2k+1)!} + \frac{C_2}{(2k-1)!} - \frac{C_4}{(2k-3)!} + \cdots + (-1)^k \frac{C_{2k-2}}{3!} + (-1)^{k-1} C_{2k} = 0 \ldots \quad (8)$$

Putting in (9), $k = 1, 2, 3, \ldots k$ and eliminating the $(k-1)$ quantities $C_2, -C_4, \ldots (-1)^k C_{2k-2}$ from the k equations thus obtained, we find

$$C_{2k} = \begin{vmatrix} \frac{2}{3!} & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \frac{4}{5!} & \frac{1}{3!} & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \frac{6}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \frac{2k}{(2k+1)!} & \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3!} \end{vmatrix} = \nabla(k), \text{ say} \ldots \quad (9)$$

Setting $k=1, 2, 3, \ldots .$ in (9) we obtain $C_2 = -\frac{1}{3}$, $C_4 = -\frac{1}{45}, \cdots$

so that $$x \cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \cdots + \nabla(k)\, x^{2k} + \cdots \quad \ldots (10)$$

4. Taking cosec x, we observe x cosec x is an even function and that $\lim_{x \to 0} (x \operatorname{cosec} x) = 1$. Hence we write

$$x \operatorname{cosec} x = 1 + P_2\, x^2 + P_4\, x^4 + \cdots + P_{2k}\, x^{2k} + \cdots \quad \ldots (11)$$

Using identity $(x \operatorname{cosec} x)\left(\frac{\sin x}{x}\right) = 1$ and proceeding as in the previous cases, we get

$$P_{2k} = \begin{vmatrix} \frac{1}{3!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2k+1)!} & \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdot & \cdots & \frac{1}{3!} \end{vmatrix} = D(k), \text{ say, } \ldots (12)$$

Putting $k=1, 2, 3, \ldots .$ we get $D_2 = \frac{1}{6}$, $D_4 = \frac{7}{360}, \ldots .$ so that

$$x \operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots + D(k)\, x^{2k} + \cdots \quad \ldots (13)$$

The corresponding series for sec x is well-known. We assume

$$\sec x = 1 + E_2\, x + E_4\, x^4 + \cdots + E_{2k}\, x^{2k} + \cdots . \quad .. (14)$$

and from the identity $\cos x \times \sec x = 1$ it follows by using the same method, we obtain the well known expression for the nth Eulerian number:

$$E_{2k} = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2k)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdot & \cdots & \frac{1}{2!} \end{vmatrix} \quad \ldots (15)$$

It is easily seen that $E_2 = 1$, $E_4 = 5$, $E_6 = 61, \cdots$ so that

$$\sec x = 1 + x^2 + 5x^4 + 61\, x^6 + \cdots + E_{2k}\, x^{2k} + \cdots \quad \ldots (16)$$

5. A comparison of the series (6), (10), and (13) we have obtained above with the expansions of the corresponding circular functions in terms of Bernoulli's Numbers, gives rise to certain expressions for Bernoulli's Numbers in the form of determinants. Thus, it is known ‡ that

$$\tan x = \sum_{1}^{\infty} (-1)^{k-1}\, 2^{2k}\, (2^{2k}-1)\, \frac{B_{2k}}{(2k)!}\, x^{2k-1} \quad \ldots \quad (17)$$

$$x \cot x = \sum_{0}^{\infty} (-1)^{k}\, 2^{2k}\, \frac{B_{2k}}{(2k)!}\, x^{2k} \quad \ldots \quad (18)$$

and since $x \operatorname{cosec} x = x \cot \frac{x}{2} - x \cot x$, we have,

$$x \operatorname{cosec} x = \sum_{0}^{\infty} (-1)^{k-1}\, (2^{2k}-2)\, \frac{B_{2k}}{(2k)!}\, x^{2k} \quad \cdot\ \cdot\ \cdot \quad \ldots \quad (19)$$

Now, comparing (17) with (6), (18) with (10), and (19) with (13), we obtain

$$B_{2k} = (-1)^{k-1} \frac{(2k)!}{2^{2k}\,(2^{2k}-1)}\, \Delta(k) \quad \ldots \quad (20);$$

$$= (-1)^{k} \frac{(2k)!}{2^{2k}}\, \nabla(k) \quad \ldots \quad (21);$$

$$= (-1)^{k-1} \frac{(2k)!}{(2^{2k}-2)}\, D(k) \quad \ldots \quad (22);$$

where $\Delta(k)$, $\nabla(k)$ and $D(k)$ are determinants of order k, defined in (5), (9) and (12) respectively.

Incidentally, from the above, we note the following relations between these three determinants ;

$$\Delta(k) : \nabla(k) : D(k) = 2^{2k}(1-2^{2k}) : 2^{2k} : (2-2^{2k}) \quad \ldots \quad (23),$$

the constant of proportionality being $(-1)^{k} \frac{B_{2k}}{(2k)!}$.

University of Travancore. K. BALACHANDRAN.

An alternate method of finding out square root of a Number

1. In this note an alternative process for obtaining square root of a number is explained. The method, though easy as it requires no multiplication at all is not always short. Subtractions are carried on after proper adjustment of digits. It has another advantage that, in case of perfect squares, the root can easily be checked.

† Vide Milne-Thomson: *Calculus of Finite Differences*, p. 148.
‡ Vide Knopp ; *Theory and Application of Infinite Series*, p. 417.

2. Let $S_n = a + (a+b) + (a+2b) \cdot \cdot \cdot \cdot \cdot \cdot + \{a + (n-1) b\}$

$$= na + \frac{n(n-1)}{2} b$$

$$= n^2 \text{ if } a = 1 \text{ and } b = 2$$

So we get, $n^2 - \{1 + 3 + 5 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot + (2n-1)\} = 0$,

or to obtain the square root of a number, the successive terms of the series 1, 3, 5, 7, ... are continually subtracted from the given number, the number of steps at which the remainder vanishes being the required root.

Check: The nth term should be $2n - 1$, i.e. the nth term should be 1 less than the double of the root.

3. Let us take as an illustration, the evaluation of $\sqrt{(729)}$.

```
7 2 9
1'
-----
6
3
-----   (Corresponds to 20 subtractions)
3 2 9
  4 1
-----
2 8 8
  4 3
-----
2 4 5
  4 5
-----
2 0 0
  4 7
-----
1 5 3
  4 9
-----
1 0 4
  5 1
-----
  5 3
  5 3
-----   The required value is 27.
```

Ths first 2 steps are equivalent to subtracting $100 + 300 = 20^2 = 1 + 3 + 5 \cdot \cdot \cdot$ (20 terms). The third step requires explanation.

As 5 cannot be subtracted from 3, 29 is carried after three. Now the 21st. (10 + 10 + 1) subtraction is reached. So the number to be subtracted is $2 \times 21 - 1$ or 41. To make the process mechanical, it

may be stated that 4 which is 1 less than the omitted term 5 of the arithmetic progression is placed below the ten's place (2) and 1 again is placed below the unit's place (9), the process of subtraction being carried on with regular increments of 2. Such are the only steps which may seem a bit complicated at the first sight, but will become easy with a little practice.

To obtain the root, the steps are counted as follows. *Two* times the digits have been subtracted continually. Next shifting has been necessary and then *seven* more continuous subtractions, the reminder vanishes. So the root is 27.

Check: $53+1=2\times 27$, which checks the result.

2·2. As another example we shall find the square root of 54856666225.

5, 4 8 5 6 6 6 6 2 2 5 1		2 steps
4 3		
1 4 8 4 1		3 steps
1 0 7 4 3		
6 4 4 5		
1 9 5 6 4 6 1		4 steps
1 4 9 5 4 6 3		
1 0 3 2 4 6 5		
5 6 7 4 6 7		

1 0 0 6 6 4 6 8 1		2 steps
5 3 8 5 4 6 8 3		
7 0 2 6 2 4 6 8 4 1		1 step
2 3 4 2 1 2 5 4 6 8 4 2 1		5 steps
1 8 7 3 7 0 4 4 6 8 4 2 3		
1 4 0 5 2 8 1 4 6 8 4 2 5		
9 3 6 8 5 6 4 6 8 4 2 7		
4 6 8 4 2 9 4 6 8 4 2 9		

Therefore the required square root is 234215.

Again, $468429+1=2\times 234215$. which shows that the root is correct.

3. A more complicated case where the given number is not a perfect square is worked out below.

Find the square root of 184·25347 upto three places of decimal.

1 8 4·2 5, 3 4, 7 0 1	1 step	2 0 0 3 4 2 7 0 1	
8 4 2 1		1 7 3 3 3 2 7 0 3	
6 3 2 3		1 4 6 3 0 2 7 0 5	
4 0 2 5	3 steps	1 1 9 2 5 2 7 0 7	
1 5·2 5 2 6 1		9 2 1 8 2 7 0 9	
1 2 6 4 2 6 3		6 5 0 9 2 7 1 1	
1 0 0 1 2 6 5		3 7 9 8 2 7 1 3	7 steps
7 3 6 2 6 7		1 0 8 5 7 9 2 7 1 4 1	
4 6 9 2 6 9	5 steps	8 1 4 2 9 2 7 1 4 3	
		5 4 2 8 6 2 7 1 4 5	
		2 7 1 4 1 2 7 1 4 7	4 steps
		etc.	

Therefore the required root = 13·574 (approximately).

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Pedal Conic

Let P and Q be two points in the plane of a triangle ABC such that the lines drawn through Q parallel to PA, PB, PC meet BC, CA, AB respectively in three collinear points. We may call Q a *pedal point* of P with respect to the triangle.

Taking ABC as the triangle of reference let the areal co-ordinates of P, Q be $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$. Then, the condition that Q may be a pedal point of P is easily seen to be $\sum \frac{1}{x_2}\left(\frac{1}{y_1}+\frac{1}{z_1}\right)=0$

which is also the condition that P may be a pedal point of Q showing that the relation between P and Q is reciprocal. Also the locus of pedal points of P is the circumconic $\sum x_1 (y_1 + z_1) yz = 0$. This conic may be called the *pedal conic* of P and the point P the *pedal point* of the circumconic. Further it is easily seen that to every circumconic there is a unique pedal point and that if the pedal conic of P passes through Q the pedal conic of Q passes through P. As a particular case we note that the pedal conic of the orthocentre is the circumcircle and that the pedal conic of any point on the circumcircle is a rectangular hyperbola.

The coordinates of the centre O of the pedal conic of $(x_1\ y_1\ z_1)$ are $\left(\frac{y_1+z_1}{2}, \frac{z_1+x_1}{2}, \frac{x_1+y_1}{2}\right)$. Hence if G be the centroid of the triangle, then GP : OG = 2 : 1 from which we infer that if a point P traces any curve, the centre of its pedal conic traces a similar curve. We may also note that given four points, the pedal conic of each of them with respect to the triangle formed by the other three is similar to the conic which is the locus of centres of conics through the four points.

K. O. AUGUSTINE.

Remarks on the above Note

Let ABC be any triangle, ω the line at infinity and P any point in the plane of the triangle. The conics through ABCP meet ω in point pairs of an involution range whose double points, say, α, β are the points at infinity on the centre locus Γ of the pencil of conics. We define the conic π through $ABC\alpha\beta$ the *pedal conic* of P with respect to ABC and show that this definition (which seems to be more natural than that given in the note mentioned above) immediately leads to the verification of the various relations shown to exist between the point P and its pedal conic π.

Among the conics through ABCP are three line pairs BC, AP; CA, BP; AB, CP whose meets with ω say L, X; M, Y; N, Z respectively, belong to the involution range and, therefore, separate harmonically the double points α, β. Hence, if Q be any point on π QL, QX etc., are parallel to conjugate diameters of π. And since QL is parallel to BC, QX is parallel to the diameter conjugate to BC. Thus QX, QY, QZ are parallel to the diameters conjugate to BC, CA, AB and, therefore,* meet BC, CA, AB respectively, in three collinear points.

* Taylor: *Geometrical conics* p. 366. The result is easily seen to be true by projecting π orthogonally into a circle.

Let Q be any point on π and Ψ its pedal conic. By the definition of the pedal conic the points at infinity on Ψ are harmonic with respect to α, β. Thus Ψ is a conic through ABCP and we have, therefore, the reciprocity relation that if the pedal conic of P passes through Q, then the pedal conic of Q passes through P. We may also note that the pedal conic of the orthocentre of the triangle is the circumcircle, follows immediately from the definition.

The other results mentioned in the note could be obtained very easily by projecting π orthogonally into a circle and using the well known properties of the circumcentre, orthocentre, centroid etc.

Annamalainagar K. RAṆGASWAMI.

A fallacy in deducing the attraction of a plane from that of a spherical shell

1. Whenever we get a result relating to a sphere of radius r, either in pure or in applied mathematics, it is not unusual, unless there is something to excite suspicion, to pass to the limit when $r \to \infty$ and thus obtain the corresponding result for a plane. Ordinarily we expect the theorem thus obtained to be correct. The example given below shows that the trouble may be much deeper than many of us are likely to suspect (unless one is of a highly suspicious temperament, which is by no means bad for his mathematics!) and should thus serve as a warning for the unwary.

Let O be the centre of a sphere of radius r and P a point distant $r+d$ from O, so that the nearest point Q of the sphere from P is at a distance d from it. When $r \to \infty$, the sphere becomes a plane distant d from P, and Q is the foot of the perpendicular from P on the plane.

The attraction at P of the spherical shell supposed to be of uniform surface density ρ is $\gamma 4\pi r^2 \rho/(r+d)^2$. When $r \to \infty$, this tends to the limit $4\pi\gamma\rho$, whereas it is known that the attraction of an infinite plane of density ρ at a point not on it is $2\pi\gamma\rho$.

The explanation of the discrepancy is quite simple. The sphere can be divided into 2 parts by the polar plane of P which exert equal attractions at P. When $r \to \infty$, the nearer segment which exerts half the attraction expands into the plane in question and hence the attraction of an infinite plane is only half of the limiting value obtained. The other segment becomes the plane at infinity

and this exerts also an attraction $2\pi\gamma\rho$ at P. In other words, the limit of the sphere is a pair of planes, as projective geometry teaches us, one of these being the plane at ∞.

2. However, the projective group has no intrinsic relation to the problem and the paradox is one which should be traced to the inverse square law, or if we prefer the physics of continuity, its differential analogue the equation of Laplace. Taking the law of attraction to be the inverse nth power of the distance, we shall find that the integral giving the attraction of the sphere converges as $r \to \infty$ only for $n \geqslant 2$. Dividing the spherical surface into two segments by the polar plane of P, we shall find that for $n > 2$, the attraction of the farther segment tends to zero with increasing r. The nearer segment expands out into infinite plane and its attraction becomes in the limit the attraction of the plane. Thus *in all cases except for $n = 2$, no fallacy arises in this problem by replacing a sphere of infinite radius by the plane which is its limit*, and we get the correct value $2\pi\gamma\rho/(n-1)d^{n-2}$. When $n = 2$, a term of order r^{2-n} which represents the attraction of the farther segment tends to a finite limit instead of zero giving a wrong value to the attraction of the plane. If we prefer to put it that way, we may say that for $n > 2$ the attraction of the plane at infinity is negligible while for $n = 2$ it has got to be taken into account.

Annamalainagar. A. NARASINGA RAO.

GLEANINGS

NUPTIAL MATHEMATICS

One night, in a month of the spring season, a certain young lady was lovingly happy with her husband on the floor of a big mansion, white like the moon, and situated in a pleasure garden with trees bent down with the load of the bunches of flowers and fruits, and reasonant with the sweet sounds of parrots, cuckoos and bees which were all intoxicated with the honey obtained from the flowers therein. Then on a love quarrel arising between the husband and the wife, that lady's necklace, made up of pearls, became sundered and fell on the floor. One third of that necklace of pearls, reached the maidservant there; 1/6 fell on the bed; then ½ of what remained and ½ of what remained thereafter, and again ½ of what remained thereafter, and so on counting 6 times in all, fell all of them everywhere; and there were found to remain (unscattered) 1,161 pearls If you know (how to work) miscellaneous problems (on fractions), give out the numerical measure of the pearls (in that necklace).

MAHAVIRACHARYA in *Ganitasarasangraha*
trans. by Prof. M. RANGACHARYA.

SOLUTIONS TO QUESTIONS

Question 1785

(V. THEBAULT, Le Mans, France):—Find three numbers A, B, C each of 4 digits such that the 16 digit number AABB may be the square of CC.

(Example: 1976 1976 8025 8025 = 4445 4445^2)

Solution by G. Venkatasubbiah.

CC = C × 10001. So $(CC)^2 = C^2 \times 10001^2$, i.e. $C^2 \times 100020001$. The multiplication may be set down as follows:—

Let $C^2 = abcdefgh$.

Multiplicand ...					*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Multiplier ...				1	0	0	0	2	0	0	0	1
					a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
	2*a*	2*b*	2*c*	2*d*	2*e*	2*f*	2*g*	2*h*				
a b c d	*e*	*f*	*g*	*h*								
A	A				B				B			

This multiplication will be correct only if $abcd + efgh = 10000$. So $abcd + efgh - 1 = 9999$. Hence $abcdefgh - 1$ i.e; $C^2 - 1$ is divisible by 9999. Now $9999 = 9 \times 11 \times 101$. Because $C+1$ and $C-1$ can have no common factor other than 2, the numbers 101, 11 and 9 must be the factors of either $C-1$ or $C+1$.

Let one of the two numbers $C-1$ and $C+1$ be *pqrs*; and let it be divisible by 101. Then $pq = rs$; i.e. the number is of the form *pqpq* If the same number is divisible by 11, *pq* must be divisible by 11. If it is divisible by 9, *pq* is divisible by 9. But 9 or 11 might be the factor of the other number. The other number is $pqpq \pm 2$. If this must be divisible by 11 or 9, $pqpq \pm 101$ must be divisible by 9 or 11; $pq \pm 1$ must be divisible by 9 or 11.

Because $C \pm 1$ is $pqpq$, C must be $pqpq \pm 1$, where either pq or $pq \pm 1$ is divisible by 11 or 9.

The different values of C satisfying these conditions are 9998, 9899, 5554, 5455, 4544 and 4445.

Hence we have the following results :—

(i) $9997\ 9997\ 0004\ 0004 = 9998\ 9998^2$
(ii) $9800\ 9800\ 0201\ 0201 = 9899\ 9899^2$
(iii) $3085\ 3085\ 6916\ 6916 = 5554\ 5554^2$
(iv) $2976\ 2976\ 7025\ 7025 = 5455\ 5455^2$
(v) $2065\ 2065\ 7936\ 7936 = 4544\ 4544^2$
(vi) $1976\ 1976\ 8025\ 8025 = 4445\ 4445^2$

Also solved by P. A. Hariharan who adopts more or less the same method and shows that the same procedure can be followed when A, B, C are each of two or three digits. Also solved by S. Venkatanarasimhan.

Question 1788

(A. E. PALKAR) :—Show that $2^{9961152}$ leaves the remainder 1374384229185 when divided by $(524287)^2$.

Solution by J. C. Auluck

If x is an odd number

$$(1+x)^x - 1 - x^2 \equiv 0 \pmod{2x^2}$$

because $(1+x)^x - 1 - x^2$ is even and is equal to

$$\frac{x\ (x-1)}{2!}\ x^2 + \frac{x\ (x-1)(x-2)}{3!}\ x^3 + \cdots + x^x \equiv 0 \pmod{x^2}$$

Hence $\dfrac{(1+x)^x}{2} = \mathrm{M}\ (x^2) + \dfrac{1+x^2}{2}$

Thus $\frac{1}{2}(1+x)^x$ leaves the remainder $\frac{1}{2}(1+x^2)$ when divided by x^2.

The result follows on putting $x = 2^{19} - 1 = 524287$, the 19th Mersenne number, which is odd.

Also solved by Messrs. P. Hariharan, S. Venkatanarasimhan, G. Srinivasan, Venkataraman and C. R Krishnamoorthi.

Question 1791

(P. KESAVA MENON) :—If through any point O three secants are drawn to meet a conic in P,P′ ; Q,Q′ ; R R′ respectively, then

$$\frac{\Delta\,\mathrm{PQR}}{\mathrm{OP\cdot OQ\cdot OR}} + \frac{\Delta\,\mathrm{P'Q'R'}}{\mathrm{OP'\cdot OQ'\cdot OR'}} = 0$$

Solution by K. Balachandran.

Taking O as the origin, let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \dots \quad (1)$$

Let OP, OQ, OR make angles θ_1, θ_2, θ_3 respectively with the x-axis.

Then it is obvious that OP, OP′ are the roots of the equation

$$r^2 (a \cos 2\theta_1 + 2h \cos\theta_1 \sin\theta_1 + b \sin^2\theta_1) + 2r (g \cos\theta_1 + h \sin\theta_1) + c = 0$$

Hence we have

$$\frac{OP + OP'}{OP \cdot OP'} = -\frac{2(g\cos\theta_1 + h\sin\theta_1)}{c} \qquad \dots \quad (2)$$

and similar equations exist for $\dfrac{OQ + OQ'}{OQ \cdot OQ'}$ and $\dfrac{OR + OR'}{OR \cdot OR'}$.

Now

$$\begin{aligned}\triangle PQR &= \triangle PQO + \triangle QRO - \triangle PRO \\ &= \tfrac{1}{2}\, OP \cdot OQ \sin(\theta_2 - \theta_1) + \tfrac{1}{2}\, OQ \cdot OR \sin(\theta_3 - \theta_2) \\ &\qquad - \tfrac{1}{2}\, OP \cdot OR \sin(\theta_3 - \theta_1)\end{aligned}$$

So that

$$\frac{2\triangle PQR}{OP \cdot OQ \cdot OR} = \frac{\sin(\theta_3 - \theta_2)}{OP} + \frac{\sin(\theta_1 - \theta_3)}{OQ} + \frac{\sin(\theta_2 - \theta_1)}{OR} \qquad \dots \quad (3)$$

Similarly $$\frac{2\triangle P'Q'R'}{OP' \cdot OQ' \cdot OR'} = \frac{\sin(\theta_3 - \theta_2)}{OP'} + \frac{\sin(\theta_1 - \theta_3)}{OQ'} + \frac{\sin(\theta_2 - \theta_1)}{OR'} \qquad \dots \quad (4)$$

Hence, by addition of (3) and (4),

$$\begin{aligned}2\left[\frac{\triangle PQR}{OP \cdot OQ \cdot OR} + \frac{\triangle P'Q'R'}{OP' \cdot OQ' \cdot OR'}\right] &= \sum \sin(\theta_3 - \theta_2)\left(\frac{OP + OP'}{OP \cdot OP'}\right) \\ &= -\frac{2}{c}\left[g\{\textstyle\sum \cos\theta_1 \sin(\theta_3 - \theta_2)\} + f\{\textstyle\sum \sin\theta_1 \sin(\theta_3 - \theta_2)\}\right]\end{aligned}$$

using (2)

$$= 0.$$

Hence the result.

Also solved by Messrs. P. Hariharan, P. S. J. Manickavasagam, J. C. Auluck, G. Srinivasan and C. Radhakrishna Rao.

QUESTIONS FOR SOLUTION

1798. (K. BALACHANDRAN):

If $\Delta(n, r) = \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \sigma_{r-1} & \sigma_{r-2} & \sigma_{r-3} & \cdots & \cdots & (r-1) \\ \sigma_r & \sigma_{r-1} & \sigma_{r-2} & \cdots & \cdots & \sigma_1 \end{vmatrix}$

where $\sigma_r = 1^r + 2^r + 3^r + \cdots + n^r$, show that

(i) $\Delta(n, n)$ is a perfect square.

(ii) $\Delta(n, n+j) \equiv 0$; $(j = 1, 2, 3, \ldots)$.

1799. (K. G. RAMANATHAN):—Prove that if l and m are positive integers and $l \geqslant m$, then

$$(3)^m \times (3)^l = (1)^{m-1}\ 0\ (9)^{l-m}\ (8)^{m-1}\ 9$$

Deduce that if $a \leqslant 9$, then

$$(a)^l\ (0)^m - (a)^m\ (0)^l = (a)^{l-m}\ (0)^m$$

Here $(p)^t$ means a number formed by $ppp \cdots$ repeated t times.

1800. (A. A. KRISHNASWAMI IYENGAR): Show that

$$\int_0^x e^{-x^2/2}\, dx > \sqrt{\left(\frac{\pi}{2}\right)}\left(1 - \tfrac{1}{2} e^{-x^2} - \tfrac{1}{2} e^{-x^2/2}\right),$$

for all real x.

1801. (P. KESAVA MENON):—If

$$f(n) = \cos^{-1} 2^{-n/2} \left({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots\right),$$

show that

$$f(p) + f(q) = f(p+q)$$